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Lindley-Chen Distribution with Applications

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Abstract

In this study, we have established a new distribution by using the Lindley generating family with baseline distribution as Chen distribution called Lindley-Chen (LC) distribution. We have illustrated some statistical properties of the model including the shapes of the probability density function (PDF), cumulative density function (CDF) and hazard rate function (HRF), quantile function also the skewness, kurtosis are discussed. We have employed three well-known estimation methods to estimate the model parameters namely the maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises (CVM) methods. We discuss maximum likelihood estimation of the distribution parameters and asymptotic confidence interval based on maximum likelihood. All the computations are performed in R software. The application of the model to a real data set is investigated and finally, we compared the goodness of fit attained by observed model via different estimation methods and we have compared with some other lifetime models.

Keywords: Lindley distribution; Chen distribution; Maximum likelihood estimation; Hazard function.

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1. Introduction

Most of the continuous probability distributions have been generated in recent decades but the real date sets related to engineering, finance, climatology, medicine, geology, biology, hydrology, ecology, reliability, life testing, and risk analysis do not provide a better fit to these distributions. So, the creation of new modified distributions seems to be necessary to address the problems in these fields. The generalized, extended, and modified distributions are created by adding one or more parameters or performing some transformation to the baseline distribution. Therefore, the new proposed distributions provide the best fit compared to the sub and competing models.

[1] has proposed a new two-parameter lifetime distribution with bathtub shaped or increasing failure rate (IFR) function. The cumulative distribution function (CDF) of Chen distribution is

$$G(x) = 1 - \exp[\lambda(1 - e^{x^\alpha})];\ \alpha, \lambda > 0,\ x > 0 \tag{1.1}$$

And its probability density function (PDF) is

$$f(x) = \alpha\lambda x^{\alpha-1} e^{x^\alpha} \exp[\lambda(1 - e^{x^\alpha})];\ \alpha, \lambda > 0,\ x > 0 \tag{1.2}$$

The motivation to extend the Chen distribution is to introduce a flexible model that has revealed the various shapes of the hazard and density functions. [2] has introduced the Markov Chain Monte Carlo methods for Bayesian inference of the Chen model. [3] have introduced the extended Chen (EC) distribution is derived from the generalized Burr-Hatke differential equation and nexus between the exponential and gamma variables. [4] introduced a new lifetime distribution with increasing, decreasing and bathtub-shaped hazard rate function which is constructed by the compounding of the Weibull and Chen distributions and is called Weibull–Chen (WC) distribution.

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The one parameter Lindley distribution was developed by [5] in the context of Bayesian statistics, as a counterexample to fiducial statistics. In recent years, many studies have been focused to obtain various modified forms of the baseline distribution using Lindley family presented by [6] with more flexible density and hazard rate functions. A detailed study on the Lindley distribution was done by [7].

A random variable X follows Lindley distribution with parameter λ and its probability density function (PDF) is given by

$$f(x)=\frac{\beta^2}{\beta+1}(1+x)e^{-\beta x};\ x>0,\beta>0 \tag{1.3}$$

And its cumulative density function (CDF) is

$$F(x)=1-\frac{1+\beta+\beta x}{1+\beta}e^{-\beta x};\ x>0,\beta>0 \tag{1.4}$$

Some of the modifications has made by [8] in the literature of Lindley distribution, which is quite similar to the exponential distribution. [9] investigated the estimation of the parameters using hybrid censored data. The estimation of the model parameters for censored samples by [10] and this distribution was applied by [11] to calculate competing risks in lifetime data.

In the theoretical distribution, [12] has introduced weighted Lindley distribution having two parameters and has shown that it is appropriate in modeling biological data for a mortality study. [13] has presented generalized Lindley, extended Lindley by [14], [15] for exponentiated power Lindley, [16] Lindley–Exponential distribution.

Also, we observed some continuous-discrete mixed approaches as [17] has defined the discrete Poisson-Lindley. [18] have introduced negative binomial Lindley distribution, the Pareto Poisson Lindley distribution by [19].

[20] has presented a new class of distributions to generate new distribution based on Lindley generator (Lindley-G) having additional shape parameter θ. The CDF and PDF of Li-G are respectively,

$$F(\xi;\theta,\lambda)=1-\left[1-G(\xi;\lambda)\right]^{\theta}\left[1-\frac{\theta}{\theta+1}\ln\bar{G}(\xi;\lambda)\right];\ \xi>0,\theta>0 \tag{1.5}$$

and

$$f(\xi;\theta,\lambda)=\frac{\theta^2}{\theta+1}g(\xi;\lambda)\left[1-G(\xi;\lambda)\right]^{\theta-1}\left[1-\ln\bar{G}(\xi;\lambda)\right];\ \xi>0,\theta>0 \tag{1.6}$$

where $g(\xi;\lambda)=\frac{dG(\xi;\lambda)}{d\xi}$, $\bar{G}(\xi;\lambda)=1-G(\xi;\lambda)$

The main objective of this work is to introduce a more flexible model by adding just one extra parameter to the Chen distribution to achieve a better fit to real data. We explore the properties of the L–C distribution and its applicability.

The arrangements of the contents of the proposed study are as follows. The Lindley Chen distribution is introduced and various mathematical and statistical properties are discussed in Section 2. We have employed three well-known estimation methods to estimate the model parameters namely the maximum likelihood estimation (MLE), least-square estimation (LSE) and Cramer-Von-Mises (CVM) methods. For the maximum likelihood estimation (MLE) procedure we have discussed the associated confidence intervals using the observed information matrix in Section 3. In Section 4, a real data set has been analyzed to explore the applications and suitability of the proposed distribution. In this section, we have illustrated the maximum likelihood (MLE), least-square (LSE) and Cramer-Von-Mises (CVM) estimates and also we have constructed the approximate confidence intervals for MLEs. Finally, in Section 5 we have presented some concluding remarks.

2. The Lindley Chen Distribution (L-C)

Taking (1.1) as a CDF of baseline distribution $G(\xi;\lambda)$ and (1.2) as PDF $g(\xi;\lambda)$, then (1.5) and (1.6) becomes

$$F(x)=1-\left[1-\lambda\left(\frac{\theta}{1+\theta}\right)\left(1-e^{x^{\alpha}}\right)\right]\exp\left[\lambda\theta\left(1-e^{x^{\alpha}}\right)\right];\ \alpha,\lambda,\theta>0,\ x>0 \tag{2.1}$$

and

$$f(x)=\alpha\lambda\left(\frac{\theta^2}{1+\theta}\right)x^{\alpha-1}e^{x^\alpha}\left\{1-\lambda\left(1-e^{x^\alpha}\right)\right\}\exp\left\{\lambda\theta\left(1-e^{x^\alpha}\right)\right\};\ \alpha,\lambda,\theta>0,\ x>0 \qquad (2.2)$$

respectively are the CDF and PDF of new purposed Lindley-Chen distribution.

2.1. Reliability/Survival function

The Reliability/Survival function of L-C distribution is

$$R(x)=1-F(x)$$

$$=\left[1-\lambda\left(\frac{\theta}{1+\theta}\right)\left(1-e^{x^\alpha}\right)\right]\exp\left[\lambda\theta\left(1-e^{x^\alpha}\right)\right];\ \alpha,\lambda,\theta>0,\ x>0 \qquad (2.3)$$

2.2. Hazard function

Suppose that x be survival time of an item and we desire the probability that it will not survive for an additional time *dx* then, hazard rate function is,

$$h(x)=\lim_{dx\to 0}\frac{pr(x\le X<x+dx)}{dx.R(x)}=\frac{f(x)}{R(x)}=\frac{f(x)}{1-F(x)};\ 0<x<\infty$$

$$=\alpha\lambda\theta^2\frac{x^{\alpha-1}e^{x^\alpha}\left\{1-\lambda\left(1-e^{x^\alpha}\right)\right\}}{1+\theta\left(1-\lambda+\lambda e^{x^\alpha}\right)};\ \ x>0 \qquad (2.4)$$

We have plotted the graph of the probability density function and hazard function of L-C distribution in Figure 1. It is found that the shapes of the Lindley Chen (L-C) density are arc, J-shaped, reverse J-shaped, negative-skewed, positive-skewed and symmetrical. The hazard rate function (HRF) for the L-C distribution is also flexible due to its various shapes such as increasing, decreasing, decreasing–increasing, increasing–decreasing and bathtub.

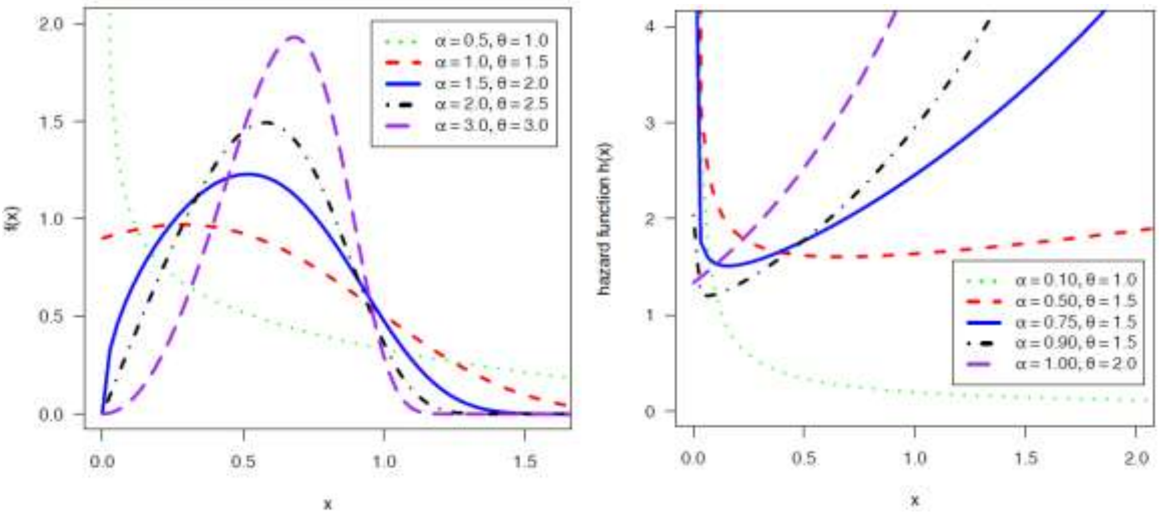

Figure 1. *Graph of PDF (left panel) and hazard function (right panel) for λ = 1 and different values of α and θ.*

2.3. Quantile function of L-C distribution

In probability and statistics, the quantile function, associated with a probability distribution of a random variable, specifies the value of the random variable such that the probability of the variable being less than or equal to that value equals the given probability. It is also called the percent-point function or inverse cumulative distribution function.

$$Q(p)=F^{-1}(p)$$

The quantile function is

$$p-1+\left[1-\lambda\left(\frac{\theta}{1+\theta}\right)\left(1-e^{x^\alpha}\right)\right]\exp\left[\lambda\theta\left(1-e^{x^\alpha}\right)\right]=0\ ;\ 0<p<1 \qquad (2.5)$$

For the generation of the random numbers of the L-C distribution, we suppose simulating values of random variable X with the CDF (2.1). Let B denote a uniform random variable in (0,1), then the simulated values of X are obtained by setting,

$$b-1+\left[1-\lambda\left(\frac{\theta}{1+\theta}\right)\left(1-e^{x^{\alpha}}\right)\right]\exp\left[\lambda\theta\left(1-e^{x^{\alpha}}\right)\right]=0,\quad 0<b<1 \qquad (2.6)$$

and solving for x.

2.4. Skewness and Kurtosis

These measures are used mostly in data analysis to study the shape of the distribution or data set. Skewness and Kurtosis based on quantile function are

$$S_k(B)=\frac{Q(0.75)+Q(0.25)-2Q(0.5)}{Q(0.75)-Q(0.25)},\text{ and}$$

Coefficient of kurtosis based on octiles given by [21] is

$$K_u(Moors)=\frac{Q(0.875)-Q(0.625)+Q(0.375)-Q(0.125)}{Q(3/4)-Q(1/4)}$$

3. Methods of Estimation

Estimation theory is a branch of statistics that deals with estimating the values of parameters based on measured empirical data that has a random component. The parameters describe an underlying physical setting in such a way that their value affects the distribution of the measured data. An estimator attempts to approximate the unknown parameters using the measurements. Commonly used estimators (estimation methods) are listed blew,

i. Maximum likelihood estimators (MLE)
ii. Bayes estimators
iii. Method of moments estimators
iv. Cramer-von Mises estimator (CVM)
v. The maximum product of spacings (MPS) method
vi. Cramér–Rao bound
vii. Least-squares estimators (LSE)
viii. Minimum mean squared error (MMSE), also known as Bayes least squared error (BLSE)
ix. Markov chain Monte Carlo (MCMC)

We have considered different estimation procedures for the unknown parameters of the L-C distribution. We introduce three types of estimators such as the maximum likelihood (MLE), ordinary least squares (LSE), Cramer-von Mises (CVM) estimators.

3.1. Maximum Likelihood Estimation (MLE)

In this section, we discuss the maximum likelihood estimators (MLE's) of the L-C distribution.

Let $\underset{\sim}{x}=(x_1,\ldots,x_n)$ be a random sample of size 'n' from L-C(α, λ, θ), then the likelihood function L(α, λ, θ/ $\underset{\sim}{x}$) can be written as,

$$L(\alpha,\lambda,\theta\,|\,\underset{\sim}{x})=\prod_{i=1}^{n}\alpha\lambda\left(\frac{\theta^2}{1+\theta}\right)x^{\alpha-1}e^{x_i^{\alpha}}\left\{1-\lambda\left(1-e^{x_i^{\alpha}}\right)\right\}\exp\left\{\lambda\theta\left(1-e^{x_i^{\alpha}}\right)\right\} \qquad (3.1)$$

Log-likelihood density of (3.1) is

$$l(\alpha,\lambda,\theta\,|\,\underset{\sim}{x})=2n\ln\theta-n\ln(1+\theta)+n\ln\alpha+n\ln\lambda+(\alpha-1)\sum_{i=1}^{n}\ln x_i+\sum_{i=1}^{n}x_i^{\alpha}+\lambda\theta\sum_{i=1}^{n}(1-e^{x_i^{\alpha}})+\sum_{i=1}^{n}\ln\left\{1-\lambda(1-e^{x_i^{\alpha}})\right\} \qquad (3.2)$$

By differentiating $l(\alpha,\lambda,\theta\,|\,\underset{\sim}{x})$ with respect to parameters and equating to zero and solving them then we obtained maximum likelihood estimators of the model.

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\ln x_i+\sum_{i=1}^{n}x_i^{\alpha}\ln x_i+\theta\lambda\sum_{i=1}^{n}e^{x_i^{\alpha}}x_i^{\alpha}\ln x_i+\lambda\ln\alpha\sum_{i=1}^{n}\frac{x_i^{\alpha}e^{x_i^{\alpha}}}{1-\lambda(1-e^{x_i^{\alpha}})}=0$$

$$\frac{\partial l}{\partial\lambda}=\frac{n}{\lambda}+\theta\sum_{i=1}^{n}(1-e^{x_i^{\alpha}})-\sum_{i=1}^{n}\frac{(1-e^{x_i^{\alpha}})}{1-\lambda(1-e^{x_i^{\alpha}})}=0$$

$$\frac{\partial l}{\partial\theta}=\frac{2n}{\theta}-\frac{n}{1+\theta}+\lambda\sum_{i=1}^{n}(1-e^{x_i^{\alpha}})=0$$

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{n}{\alpha^2} + \sum_{i=1}^{n} x_i^{\alpha} (\ln x_i)^2 - \lambda\theta \sum_{i=1}^{n} x_i^{\alpha} e^{x_i^{\alpha}} (1 + \ln x_i) + \lambda \sum_{i=1}^{n} \frac{x_i^{\alpha} e^{x_i^{\alpha}} (\ln x_i)^2}{\left\{1-\lambda(1-e^{x_i^{\alpha}})\right\}}$$

$$+\lambda \sum_{i=1}^{n} \frac{x_i^{2\alpha} e^{x_i^{\alpha}} (\ln x_i)^2}{\left\{1-\lambda(1-e^{x_i^{\alpha}})\right\}} - \lambda^2 \sum_{i=1}^{n} \frac{x_i^{2\alpha} e^{2x_i^{\alpha}} (\ln x_i)^2}{\left\{1-\lambda(1-e^{x_i^{\alpha}})\right\}^2}$$

$$\frac{\partial^2 l}{\partial \lambda^2} = -\frac{n}{\lambda^2} - \sum_{i=1}^{n} \frac{(1-e^{x_i^{\alpha}})^2}{\left\{1-\lambda(1-e^{x_i^{\alpha}})\right\}^2}$$

$$\frac{\partial^2 l}{\partial \theta^2} = \frac{2n}{\theta} - \frac{n}{1+\theta} + \lambda \sum_{i=1}^{n} (1-e^{x_i^{\alpha}})$$

$$\frac{\partial^2 l}{\partial \alpha \partial \lambda} = -\theta \sum_{i=1}^{n} x_i^{\alpha} e^{x_i^{\alpha}} \ln x_i + \sum_{i=1}^{n} \frac{x_i^{\alpha} e^{x_i^{\alpha}} \ln x_i}{\left\{1-\lambda(1-e^{x_i^{\alpha}})\right\}} - \lambda \sum_{i=1}^{n} \frac{x_i^{\alpha} e^{x_i^{\alpha}} \ln x_i (e^{x_i^{\alpha}} - 1)}{\left\{1-\lambda(1-e^{x_i^{\alpha}})\right\}^2}$$

$$\frac{\partial^2 l}{\partial \alpha \partial \theta} = -\lambda \sum_{i=1}^{n} x_i^{\alpha} e^{x_i^{\alpha}} \ln x_i$$

$$\frac{\partial^2 l}{\partial \lambda \partial \theta} = n - \sum_{i=1}^{n} e^{x_i^{\alpha}}$$

Manually it is not possible to solve these nonlinear equations so we can use iterative techniques such as the Newton-Raphson algorithm to calculate the estimated value of the parameters. The Optim() function in R software can be used to solve them numerically.

Let us denote the parameter vector by $\underline{\psi} = (\alpha, \lambda, \theta)$ and the corresponding MLE of $\underline{\psi}$ as $\hat{\underline{\psi}} = (\hat{\alpha}, \hat{\lambda}, \hat{\theta})$, then the asymptotic normality results in, $\left(\hat{\underline{\psi}} - \underline{\psi}\right) \rightarrow N_3\left[0, \left(I\left(\underline{\psi}\right)\right)^{-1}\right]$ where $I\left(\underline{\psi}\right)$ is the Fisher's information matrix given by,

$$I\left(\underline{\psi}\right) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial \alpha \partial \lambda}\right) & E\left(\frac{\partial^2 l}{\partial \alpha \partial \theta}\right) \\ E\left(\frac{\partial^2 l}{\partial \lambda \partial \alpha}\right) & E\left(\frac{\partial^2 l}{\partial \lambda^2}\right) & E\left(\frac{\partial^2 l}{\partial \lambda \partial \theta}\right) \\ E\left(\frac{\partial^2 l}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 l}{\partial \lambda \partial \theta}\right) & E\left(\frac{\partial^2 l}{\partial \theta^2}\right) \end{pmatrix}$$

In practice, it is useless that the MLE has asymptotic variance $\left(I\left(\underline{\psi}\right)\right)^{-1}$ because we don't know $\underline{\psi}$. Hence we approximate the asymptotic variance by plugging in the estimated value of the parameters.

The common procedure is to use observed fisher information matrix O($\hat{\underline{\psi}}$) as an estimate of the information matrix $I\left(\underline{\psi}\right)$ given by

$$O\left(\hat{\underline{\psi}}\right)=-\begin{pmatrix}\frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\lambda} & \frac{\partial^2 l}{\partial\alpha\partial\theta}\\ \frac{\partial^2 l}{\partial\alpha\partial\lambda} & \frac{\partial^2 l}{\partial\lambda^2} & \frac{\partial^2 l}{\partial\lambda\partial\theta}\\ \frac{\partial^2 l}{\partial\alpha\partial\theta} & \frac{\partial^2 l}{\partial\lambda\partial\theta} & \frac{\partial^2 l}{\partial\theta^2}\end{pmatrix}_{\big|(\hat{\alpha},\hat{\lambda},\hat{\theta})} = -H\left(\underline{\psi}\right)_{\big|(\underline{\delta}=\hat{\underline{\delta}})}$$

Where H is the Hessian matrix.
The Newton-Raphson algorithm to maximize the likelihood produces the observed information matrix. Therefore, the variance-covariance matrix is given by,

$$\left[-H\left(\underline{\psi}\right)_{\big|(\underline{\delta}=\hat{\underline{\delta}})}\right]^{-1}=\begin{pmatrix}\operatorname{var}(\hat{\alpha}) & \operatorname{cov}(\hat{\alpha},\hat{\lambda}) & \operatorname{cov}(\hat{\alpha},\hat{\theta})\\ \operatorname{cov}(\hat{\alpha},\hat{\lambda}) & \operatorname{var}(\hat{\lambda}) & \operatorname{cov}(\hat{\theta},\hat{\lambda})\\ \operatorname{cov}(\hat{\alpha},\hat{\theta}) & \operatorname{cov}(\hat{\theta},\hat{\lambda}) & \operatorname{var}(\hat{\theta})\end{pmatrix}$$

Hence from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for α, λ, θ can be constructed as,

$$\hat{\alpha}\pm Z_{\alpha/2}SE(\hat{\alpha})\ ,\ \hat{\lambda}\pm Z_{\alpha/2}SE(\hat{\lambda})\ \text{and}\ \hat{\theta}\pm Z_{\alpha/2}SE(\hat{\theta}),$$

where $Z_{\alpha/2}$ is the upper percentile of standard normal variate.

3.2. Method of Least-Square Estimation (LSE)

The least-square estimators and weighted least square estimators were proposed by [22] to estimate the parameters of Beta distributions. In this article, the same technique is applied to the L-C distribution. The least-square estimators of the unknown parameters α, λ, and θ of L-C distribution can be obtained by minimizing

$$\Upsilon\left(X;\alpha,\lambda,\theta\right)=\sum_{j=1}^{n}\left[G(X_j)-\frac{j}{n+1}\right]^2 \tag{3.2.1}$$

with respect to unknown parameters α, λ, and θ.

Suppose $\Upsilon(X_{(j)})$ denotes the distribution function of the ordered random variables $X_{(1)} < X_{(2)} < \ldots\ldots < X_{(n)}$, where $\{X_1, X_2, \ldots\ldots, X_n\}$ is a random sample of size n from a distribution function G(.). Here, the least square estimators of α, λ, and θ say $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\theta}$ respectively, can be obtained by minimizing

$$\Upsilon\left(X;\hat{\alpha},\hat{\lambda},\hat{\theta}\right)=\sum_{j=1}^{n}\left[1-\left[1-\lambda\left(\frac{\theta}{1+\theta}\right)\left(1-e^{x_j^{\alpha}}\right)\right]\exp\left[\lambda\theta\left(1-e^{x_j^{\alpha}}\right)\right]-\frac{j}{n+1}\right]^2 \tag{3.2.2}$$

with respect to α, λ, and θ.

Let $U_j=\exp\left[\lambda\theta\left(1-e^{x_j^{\alpha}}\right)\right]$ and $V_j=\left(1-e^{x_j^{\alpha}}\right)$ then after differentiation with respect to α, λ, and θ we get the three nonlinear equations as

$$\frac{\partial\Upsilon\left(X;\alpha,\lambda,\theta\right)}{\partial\alpha}=2\sum_{j=1}^{n}\left[1-\left[1-\lambda\left(\frac{\theta}{1+\theta}\right)V_j\right]U_j-\frac{j}{n+1}\right]$$
$$\left\{-\lambda\theta x_j^{\alpha}\ln(x_j)\exp\left[\lambda\theta V_j+x_j^{\alpha}\right]-\frac{\lambda\theta x_j^{\alpha}\ln(x_j)\exp\left[\lambda\theta V_j+x_j^{\alpha}\right]\left[\lambda\theta e^{x_j^{\alpha}}-\lambda\theta-1\right]}{1+\theta}\right\}$$

$$\frac{\partial\Upsilon(X;\alpha,\lambda,\theta)}{\partial\lambda}=2\sum_{j=1}^{n}\left[1-\left[1-\lambda\left(\frac{\theta}{1+\theta}\right)V_j\right]U_j-\frac{j}{n+1}\right]\left\{\theta U_jV_j+\frac{\theta U_jV_j\left[\left(\theta e^{x_j^{\alpha}}-\theta\right)\lambda-1\right]}{1+\theta}\right\}$$

$$\frac{\partial\Upsilon(X;\alpha,\lambda,\theta)}{\partial\theta}=2\sum_{j=1}^{n}\left[1-\left[1-\lambda\left(\frac{\theta}{1+\theta}\right)V_j\right]U_j-\frac{j}{n+1}\right]\left\{\lambda U_jV_j+\frac{\lambda U_jV_j\left[\lambda\theta^2V_j+\lambda\theta V_j+1\right]}{(1+\theta)^2}\right\}$$

The weighted least square estimators of the unknown parameters can be obtained by minimizing

$$\Upsilon(X;\alpha,\lambda,\theta)=\sum_{j=1}^{n}w_j\left[G(X_j)-\frac{j}{n+1}\right]^2$$

with respect to α, λ, and λ. The weights w_j are $w_j=\frac{1}{V(X_{(j)})}=\frac{(n+1)^2(n+2)}{j(n-j+1)}$

Hence, the weighted least square estimators of α, λ, and θ respectively, can be obtained by minimizing,

$$\Upsilon\left(X;\hat{\alpha},\hat{\lambda},\hat{\theta}\right)=\sum_{j=1}^{n}\frac{(n+1)^2(n+2)}{j(n-j+1)}\left[1-\left[1-\lambda\left(\frac{\theta}{1+\theta}\right)\left(1-e^{x_j^{\alpha}}\right)\right]\exp\left[\lambda\theta\left(1-e^{x_j^{\alpha}}\right)\right]-\frac{j}{n+1}\right]^2 \quad (3.2.3)$$

with respect to α, λ, and θ.

3.3. Method of Cramer-Von-Mises (CVM)

We interested in Cramér-von-Mises type minimum distance estimators [23] because it provides empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. The CVM estimators of α, λ, and θ are obtained by minimizing the function

$$C(\alpha,\lambda,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{i:n}\mid\alpha,\lambda,\theta\right)-\frac{2i-1}{2n}\right]^2$$

$$C(\alpha,\lambda,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[1-\left[1-\lambda\left(\frac{\theta}{1+\theta}\right)\left(1-e^{x_j^{\alpha}}\right)\right]\exp\left[\lambda\theta\left(1-e^{x_j^{\alpha}}\right)\right]-\frac{2i-1}{2n}\right]^2 \quad (3.3.1)$$

Let $U_j=\exp\left[\lambda\theta\left(1-e^{x_j^{\alpha}}\right)\right]$ and $V_j=\left(1-e^{x_j^{\alpha}}\right)$ then after differentiation with respect to α, λ, and θ we get the three nonlinear equations as

$$\frac{\partial C}{\partial\alpha}=2\lambda\frac{\theta^2}{1+\theta}\sum_{j=1}^{n}\left[1-\left[1-\lambda\left(\frac{\theta}{1+\theta}\right)V_j\right]U_j-\frac{2i-1}{2n}\right]\left\{x_j^{\alpha}\ln(x_j)\exp\left[\lambda\theta V_j+x_j^{\alpha}\right]\right\}\left\{\lambda-1-\lambda e^{x_j^{\alpha}}\right\}$$

$$\frac{\partial C}{\partial\lambda}=2\frac{\theta}{1+\theta}\sum_{j=1}^{n}U_jV_j\left[1-\left[1-\lambda\left(\frac{\theta}{1+\theta}\right)V_j\right]U_j-\frac{2i-1}{2n}\right]\left\{\theta+\left(\theta e^{x_j^{\alpha}}-\theta\right)\lambda\right\}$$

$$\frac{\partial C}{\partial\theta}=\frac{2\lambda}{(1+\theta)^2}\sum_{j=1}^{n}U_jV\left[1-\left[1-\lambda\left(\frac{\theta}{1+\theta}\right)V_j\right]U_j-\frac{j}{n+1}\right]\left\{(1+\theta)^2+\lambda\theta^2V_j+\lambda\theta V_j+1\right\}$$

Equating them to zero and solving simultaneously we get CVM estimators.

4. Real Data Analysis and Result

In this section, we fit our model on the data of tensile strength of 65 observations of failure stresses of single carbon fibers of length 50 mm. The data were also used by [24] and later discussed in [25]. The data were as follows

1.339, 1.434, 1.549, 1.574, 1.589, 1.613, 1.746, 1.753, 1.764, 1.807, 1.812, 1.84, 1.852, 1.852, 1.862, 1.864, 1.931, 1.952, 1.974, 2.019, 2.051, 2.055, 2.058, 2.088, 2.125, 2.162, 2.171, 2.172, 2.18, 2.194, 2.211, 2.27, 2.272, 2.28, 2.299, 2.308, 2.335, 2.349, 2.356, 2.386, 2.39, 2.41, 2.43, 2.431, 2.458, 2.471, 2.497, 2.514, 2.558, 2.577, 2.593, 2.601, 2.604, 2.62, 2.633, 2.67, 2.682, 2.699, 2.705, 2.735, 2.785, 3.02, 3.042, 3.116, 3.174

By using the likelihood function in (3.2), we have computed the maximum likelihood estimates directly by using [26]. From the above data set, we have obtained $\hat{\alpha}$ = 1.2681, $\hat{\lambda}$ = 28.9639 and $\hat{\theta}$ = 0.00355 corresponding Log-Likelihood value is -37.0209. In Table 1 we have demonstrated the MLE's with their standard errors (SE) and 95% confidence interval for α, λ and θ.

Table 1. MLE, SE and 95% confidence interval

Parameter	MLE	SE	95% ACI
Alpha	1.2681	0.05867	(1.1532, 1.3831)
Lambda	28.9639	5.157	(18.8562, 39.0716)
Theta	0.00355	0.00096	(0.00167, 0.00543)

The Profile log-likelihood functions of parameters α, λ and θ are displayed in Figure 2. It is revealed that the estimated parameters using the MLE method are unique.

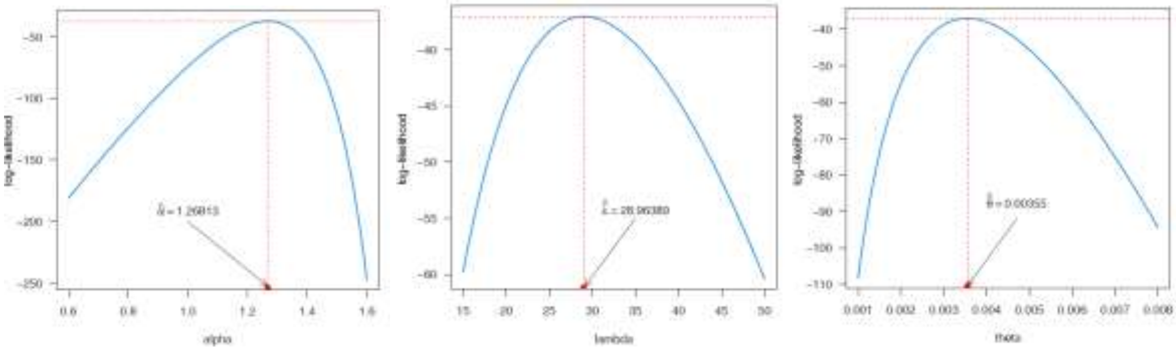

Figure 2. *Profile log-likelihood functions of α, λ and θ.*

Table 2. Estimated parameters, log-likelihood, AIC, BIC and AICC

Method	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	-LL	AIC	BIC	AICC
MLE	1.2681	1.2681	1.2681	37.0209	80.0418	86.5649	80.4352
LSE	1.2897	1.3012	1.4891	37.2314	80.3251	86.5901	80.5028
CVE	1.362	1.2901	2.5220	37.5241	80.1328	86.5887	80.5422

To assess the goodness of fit of any distribution we used to plot the graphs of PDF and CDF respectively. To know more about the nature of the distribution we have to plot Q-Q and P-P plots. In particular, the Q-Q plot is used widely it provides more information about the lack-of-fit at the tails of the distribution, whereas the P-P plot emphasizes the lack-of-fit. In Figure 3 we have presented the graph of Q-Q and CDF of L-C distribution. From Figure 3 it is proven that the L-C model fits the data properly.

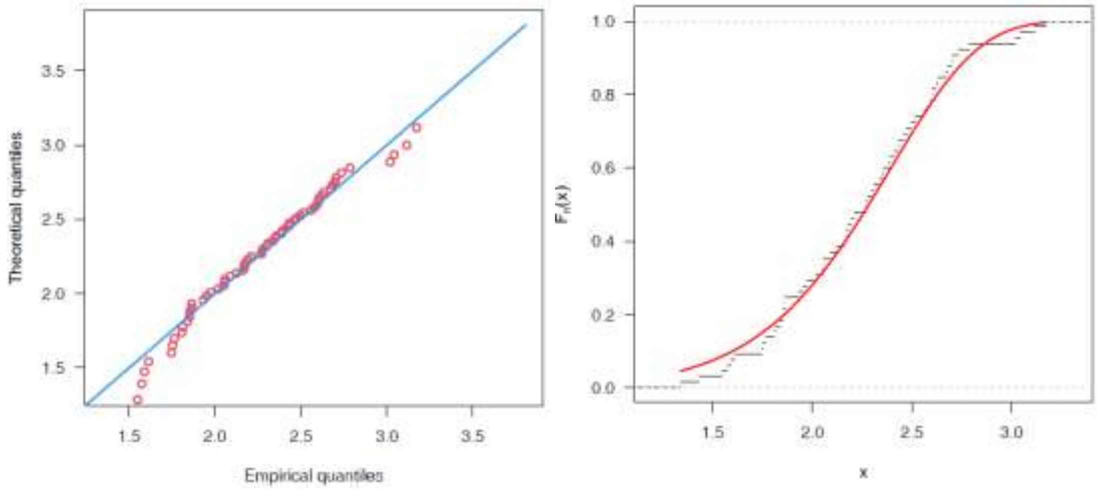

Figure 3. *The graph of the Q-Q plot (left panel) and CDF plot (right panel).*

To illustrate the goodness of fit of the Lindley inverse Weibull distribution, we have taken some well known distribution for comparison purpose which are listed blew,

I. Gompertz distribution (GZ):

The probability density function of Gompertz distribution [27] with parameters α and θ is

$$f_{GZ}\left(x\right)=\theta\ e^{\alpha x}\ exp\left\{\frac{\theta}{\alpha}\left(1-e^{\alpha x}\right)\right\}\quad ;x\geq 0,\theta>0,-\infty<\alpha<\infty.$$

II. Exponential power (EP) distribution:

The probability density function Exponential power (EP) distribution [28] is

$$f_{EP}(x)=\alpha\,\lambda^{\alpha}\ x^{\alpha-1}\ e^{(\lambda\, x)^{\alpha}}\ \exp\left\{1-e^{(\lambda\, x)^{\alpha}}\right\}\quad ;(\alpha,\lambda)>0,\quad x\geq 0.$$

where α and λ are the shape and scale parameters, respectively.

III. Chen distribution:

[1] has introduced Chain distribution having probability density function (PDF) as

$$f\left(x;\lambda,\theta\right)=\lambda\beta x^{\theta-1}e^{x\theta}\ exp\left\{\lambda\left(1-e^{x^{\theta}}\right)\right\}\quad ;(\lambda,\theta)>0,x>0.$$

IV. The inverse Weibull (IW) distribution

The probability density function (PDF) of a random variable X of IW [29] is given by

$$g\left(x\right)=\alpha\beta x^{-(\beta+1)}\exp\left(-\alpha x^{-\beta}\right);\ \ x\geq 0,\ \alpha>0,\ \beta>0$$

For the evaluation of potentiality of the LC distribution, we have calculated the Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) which are presented in Table 3.

Table 3. Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	-LL	AIC	BIC	CAIC	HQIC
LC	37.0209	80.0418	86.5649	80.4352	82.6156
Gompertz	38.9102	81.8205	86.1692	82.0140	83.5363
EP	38.9455	81.8909	86.2397	82.0784	83.6068
Chen	40.5975	85.1949	89.5437	85.3885	86.9108
IW	43.8600	91.7200	96.0688	91.9136	93.4359

The Histogram and the density function of fitted distributions and Empirical distribution functions with estimated distribution function of LC distribution and some selected distributions are presented in Figure 4.

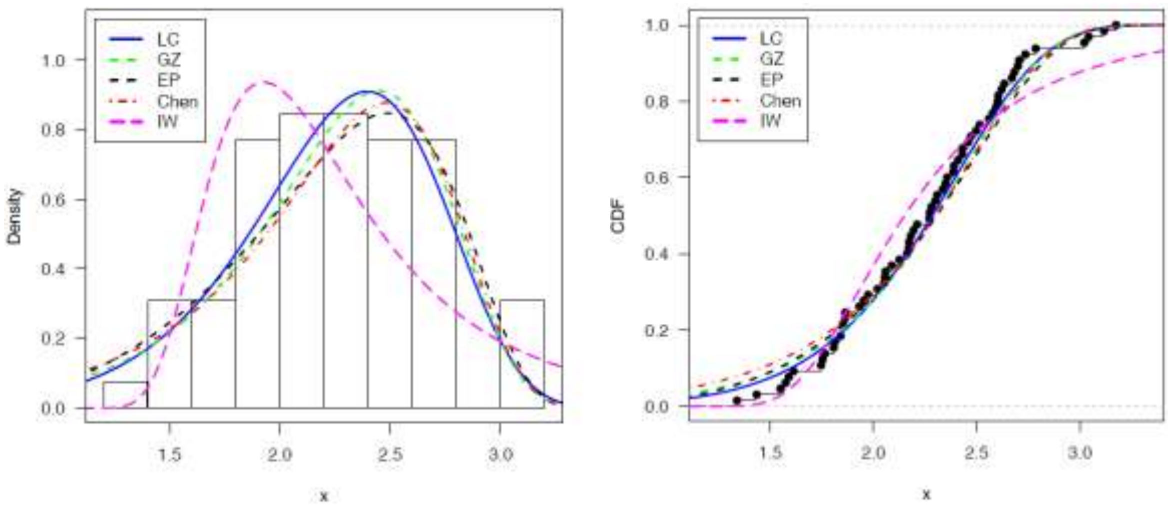

Figure 4. *The Histogram and the density function of fitted distributions (left panel) and Empirical distribution function with estimated distribution function (right panel).*

To compare the goodness-of-fit of the LC distribution with other competing distributions we have presented the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics in Table 4. It is observed that the LC distribution has the minimum value of the test statistic and higher *p*-value thus we conclude that the LC distribution gets quite better fit and more consistent and reliable results as compared to others.

Table 4. The goodness-of-fit statistics and their corresponding p-value

Model	***KS(p-value)***	***AD(p-value)***	***CVM(p-value)***
LC	0.0571(0.9838)	0.0432(0.9172)	0.4653(0.7817)
Gompertz	0.0697(0.9107)	0.0799(0.6940)	0.7486(0.5191)
EP	0.0807(0.7910)	0.1202(0.4957)	0.9623(0.3773)
Chen	0.0911(0.6540)	0.1141(0.5213)	1.0414(0.3360)
IW	0.1250(0.2618)	0.2653(0.1699)	1.6798(0.1389)

5. Conclusions

We have introduced a new three-parameter Lindley Chen (L-C) distribution, which is the extension of the Chen distribution. Actually, the L-C distribution is motivated by the extensive use of the Chen distribution in many applied fields and further its generalization provides more flexibility in the analysis of real data. We have provided the PDF, the CDF, and the shapes of the hazard function. The shape of the probability density function of the L-C distribution is unimodel and positively skewed, while the hazard function of the L-C distribution is increasing, decreasing, decreasing–increasing, increasing–decreasing and bathtub. We have employed three well-known estimation methods to estimate the model parameters namely the maximum likelihood estimation (MLE), least-square estimation (LSE) and Cramer-Von-Mises (CVM) methods. Finally, a real data set is used to investigate the applicability of the proposed model. It is concluded that the proposed model is more flexible and provide a better fit for survival data as compared to some other models.

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